Efficient Computation of the Inomplete Gamma and Beta Ratio Functions

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Abstract

In this document we summarize and prove some results around the gamma function, the incomplete gamma ratio function, the beta function and the incomplete beta ratio function. These results are used in the numerical evaluation of these functions in the STK++ project.

1 The Gamma function and related functions

1.1 The Gamma (factorial) function

The gamma function is given by the euler's integral

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt \tag{1}$$

with $a \neq 0, -1, -2, \ldots$. We have the well known identity $\Gamma(a+1) = a!$ for $a \in \mathbb{N}$.

Serie of the Gamma function It can be computed using the following development du to lanczos (see [Lan64], [Pug04])

$$\Gamma(z+1) = \sqrt{2\pi}\left(z+g+\frac{1}{2}\right)^{z+\frac{1}{2}} e^{-\left(z+g+\frac{1}{2}\right)} A_g(z) \tag{2}$$

with

$$A_g(z) = c_0 + \sum_{k=1}^{N} \frac{c_k(g)}{z+k}.$$

The coefficients $c_k(g)$ have a closed (but complicated) form and can be calculated explicitly.

Asymptotic expansion of the Log-Gamma function The asymptotic expansion of the logarithm is also called "Stirling's series":

$$\ln(\Gamma(z+1)) = z\ln z - z + \frac{1}{2}\ln(2\pi z) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \ldots \tag{3}$$

Asymptotic expansion of the Gamma function In this part we will find the Stirling's formula which give an asymptotic formula for the Gamma function.

For any z such that $0 \leq z$

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^z dt = z^{z+1} e^{-z} \int_{-1}^{+\infty} e^{-z(s-\log(s+1))} ds$$

with $t = z(s-1)$. Using the change of variable $u^2/2 = s - \log(s+1)$, with boundary conditions

- $s = -1$ corresponds to $u = -\infty$
- $s = 0$ correspons to $u = 0$
- $s = +\infty$ corresponds to $u = +\infty$

we get

$$\Gamma(z+1) = z^{z+1} e^{-z} \int_{-\infty}^{+\infty} e^{-z\frac{u^2}{2}} s'(u) du. \tag{4}$$

Writing $s(u) = \sum_{k=0}^{+\infty} a_k u^k$, from $ss' = (1+s)u$, we get $a_0 = 0$, $a_1 = 1$, and for $k > 1$

$$a_k = \frac{a_{k-1}}{k+1} - \frac{1}{k+1} \sum_{j=2}^{k-1} j a_j a_{k-j+1} \tag{5}$$

Recursion (5) give that

$$a_2 = \frac{1}{3}, \quad a_3 = \frac{1}{36}, \quad a_4 = -\frac{1}{270}, \quad a_5 = \frac{1}{4320}, \quad a_6 = \frac{1}{17010}, \quad \ldots$$

therefore

$$s'(u) = \sum_{k=1}^{+\infty} k a_k u^{k-1} = 1 + \frac{2}{3} u + \frac{1}{12} u^2 - \frac{2}{135} u^3 + \frac{1}{864} u^4 + \frac{1}{2835} u^5 + \ldots \tag{6}$$

Inserting (6) in (4) we get

$$\begin{aligned} \Gamma(z+1) &= z^{z+1} e^{-z} \sum_{k=1}^{+\infty} k a_k \int_{-\infty}^{+\infty} u^{k-1} e^{-z\frac{u^2}{2}} \, du \\ &= \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \sum_{k=1}^{+\infty} k a_k \left(\frac{1}{\sqrt{z}}\right)^{k-1} \int_{-\infty}^{+\infty} u^{k-1} \phi(u) \, du \end{aligned}$$

and using (25) we obtain the famous Stirling's formula

$$\begin{aligned} \Gamma(z+1) &= \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \sum_{n=0}^{+\infty} (2n+1)!! \frac{a_{2n+1}}{z^n} \qquad (7) \\ &= \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51\ 840 z^3} - \frac{571}{2\ 488\ 320 z^4} + \ldots\right) \end{aligned}$$

1.2 The incomplete Gamma functions

The incomplete gamma functions are given by

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt, \quad a > 0, x > 0 \tag{8}$$

and

$$\Gamma(a,x) = \int_x^{+\infty} e^{-t} t^{a-1} dt, \quad a > 0, x > 0. \tag{9}$$

The incomplete gamma ratio functions are given by

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt = \frac{\gamma(a,x)}{\Gamma(a)}, \quad a > 0, x > 0 \tag{10}$$

and

$$Q(a,x) = \frac{1}{\Gamma(a)} \int_x^{+\infty} e^{-t} t^{a-1} dt = \frac{\Gamma(a,x)}{\Gamma(a)}, \quad a > 0, x > 0. \tag{11}$$

Clearly $P + Q = 1$.

1.2.1 Series of the incomplete Gamma ratio function $Q(a,x)$

Integrating by part in (11) it is easely show that

$$Q(b+n,x) = \frac{e^{-x} x^{b+n-1}}{\Gamma(b+n)} + Q(b+n-1,x)$$

Integrating by part n times, we get the relation

$$Q(b+n,x) = \frac{e^{-x} x^{b+n-1}}{\Gamma(b+n)} \sum_{k=0}^{n-1} \frac{\Gamma(b+n)}{\Gamma(b+n-k)} \left(\frac{1}{x}\right)^k + Q(b,x) \tag{12}$$

Thus if $a = b+n$ with $0 < b \leq 1$ and x are sufficiently large, we can neglect the residual term $Q(b,x)$ and evaluate the gamma ratio function Q using the serie in (12).

1.2.2 Asymptotic expansion of the incomplete Gamma ratio function $Q(a,x)$

This part is a rewritting of the document `http://members.aol.com/iandjmsmith/PoissonApprox.htm`, see ([Smi]). The aim is to find an asymptotic expansion of the gamma ratio function $Q(a,x)$ for large values of a.

Using the same change of variable defined in 1.1 we get that for any d

$$\Gamma(z+1, z+d) = z^{z+1} e^{-z} z \int_B^{+\infty} e^{-z\frac{u^2}{2}} s'(u)\, du, \tag{13}$$

with $B = \sqrt{-2(\log(1+d/z) - d/z}$ if $d > 0$ and $B = -\sqrt{-2(\log(1+d/z) - d/z}$ if $d < 0$.

Writting $D = \sqrt{z}B$, we get together with (6) and (5)

$$\begin{aligned}
\Gamma(z+1, z+d) &= \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \sum_{k=1}^{+\infty} k a_k \left(\frac{1}{\sqrt{z}}\right)^{k-1} \int_D^{+\infty} u^{k-1}\phi(u)\,du \\
&= \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \sum_{n=0}^{+\infty} (2n+1)a_{2n+1} \left(\frac{1}{\sqrt{z}}\right)^{2n} \Phi_{2n}(-D) \\
&- \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \sum_{n=0}^{+\infty} (2n-2)a_{2n+2} \left(\frac{1}{\sqrt{z}}\right)^{2n+1} \Phi_{2n+1}(-D)
\end{aligned}$$

where $\Phi_n(z)$ is defined in the appendix (26).

Using the recurrence formulas (27) we get the expansion

$$\begin{aligned}
\Gamma(z+1, z+d) &= \frac{\sqrt{2\pi z}}{2}\left(\frac{z}{e}\right)^z \Phi(-D) \left[\sum_{n=0}^{+\infty} \frac{(2n+1)!!}{z^n} a_{2n+1}\right] \\
&+ \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \phi(D) \left[\sum_{n=1}^{+\infty} \frac{(2n+1)!!}{z^n} a_{2n+1} \sum_{j=1}^{n} \frac{D^{2j-1}}{(2j-1)!!}\right] \\
&+ \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \phi(D) \frac{1}{\sqrt{z}} \left[\sum_{n=0}^{+\infty} \frac{(2n+2)!!}{z^n} a_{2n+2} \sum_{j=0}^{n} \frac{D^{2j}}{(2j)!!}\right]
\end{aligned}$$

Using the Stirling's formula (7), noting $c_n = n!!a_n$ and $C = D^2/2$, we get

$$\begin{aligned}
Q(x+1, x+d) &= \Phi(-D)D\phi(D) \frac{\sum_{n=1}^{+\infty} \frac{c_{2n+1}}{z^n} a_n(C)}{\sum_{n=0}^{+\infty} \frac{c_{2n+1}}{z^n}} \\
&+ \frac{\phi(D)}{\sqrt{z}} \frac{\sum_{n=0}^{+\infty} \frac{c_{2n}}{z^n} b_n(C)}{\sum_{n=0}^{+\infty} \frac{c_{2n+1}}{z^n}}
\end{aligned}$$

where the coefficients $b_j(C)$ and $a_j(C)$ are defined by the following recurrence formulas

$$\begin{aligned}
a_1(C) = 1, \qquad a_n(C) &= a_{n-1}(C) + \frac{C^{n-1}}{(1+1/2)\ldots(n-1/2)} \\
b_0(C) = 1, \qquad b_n(C) &= b_{n-1}(C) + \frac{C^n}{n!}
\end{aligned}$$

2 The Beta function and related functions

2.1 The Beta function

The Beta function is defined as

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \qquad (14)$$

with $a > 0$ and $b > 0$.

2.1.1 Expansion of the Beta function

Our aim is to develop a saddle-point expansion for the Beta function. Let $p = a/(a+b)$ and $q = 1-p = b/(a+b)$, then

$$\begin{aligned} B(a+1,b+1) &= \int_0^1 t^a(1-t)^b dt \\ &= p^a q^b \int_0^1 \left[\left(\frac{t}{p}\right)^p \left(\frac{1-t}{q}\right)^q\right]^{a+b} dt \end{aligned}$$

We will now transform the integration variable t into a new variable u by the implicit transcendental equation

$$e^{-\frac{u^2}{2}} = \left(\frac{t}{p}\right)^p \left(\frac{1-t}{q}\right)^q \tag{15}$$

with boundary conditions:

- $t = 1$ corresponds to $u = +\infty$,
- $t = p$ corresponds to $u = 0$.
- $t = 0$ corresponds to $u = -\infty$,

Then B appears in the form

$$B(a+1,b+1) = p^a q^b \int_{-\infty}^{+\infty} e^{-(a+b)\frac{u^2}{2}} t'(u)du. \tag{16}$$

Now the implicit equation (15) can be changed to the non-linear differential equation

$$tt' - pt' - ut + ut^2 = 0. \tag{17}$$

with the boundary condition

$$t(0) = p.$$

By putting the formal expansion $t(u) = p + a_1 u + a_2 u^2 + \dots$ in the differential equation (17) we get the following recurrence relation for the a_n, $n \geq 1$

$$\sum_{k=0}^{n-1} \left[(k+1)a_{k+1}a_{n-k} + a_k a_{n-1-k}\right] - a_{n-1} = 0$$

This can be rewritten

$$(n+2)\sqrt{pq}a_{n+1} = (q-p)a_n - \sum_{k=1}^{n-1} \left[(k+1)a_{k+1}a_{n+1-k} + a_k a_{n-k}\right].$$

Solving in succession we get

$$a_1 = \sqrt{pq}, \quad a_2 = \frac{q-p}{3}, \quad a_3 = \frac{1}{4\sqrt{pq}}\left[\left(\frac{q-p}{3}\right)^2 - pq\right], \quad \dots$$

and thus

$$t'(u) = \sqrt{pq} + 2\frac{2p+1}{3}u + 3a_3u^2 + 4a_4u^3 + \dots \tag{18}$$

Substituting the expansion 18 in the equation (16), and using 25 we get

$$\begin{aligned} B(a+1,b+1) &= p^a q^b \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(2k+1)!}{2^k k!} \left(\frac{1}{a+b}\right)^{\frac{2k+1}{2}} a_{2k+1} \\ &= p^a q^b \sqrt{\frac{2\pi}{a+b}} \sum_{k=0}^{\infty} (2k+1)!! \left(\frac{1}{a+b}\right)^{k} a_{2k+1} \end{aligned}$$

2.2 The Incomplete Beta functions

The incomplete Beta function is given by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt \tag{19}$$

with $x > 0$ and the incomplete Beta ratio function is given by

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}. \tag{20}$$

The incomplete Beta ratio function satisfy the following symetry formula

$$I_x(a,b) = 1 - I_{1-x}(b,a). \tag{21}$$

2.2.1 Serie representation of the Beta function

Replacing in 20 the second factor in the integrand with it's binomial expansion The Beta ratio function we get the well known series represenation

$$I_x(a,b) = \frac{x^a}{B(a,b)} \cdot \left(\frac{1}{a} + \sum_{n=1}^{\infty} \frac{(1-b)(2-b)\dots(n-b)}{n!(a+n)} \right) \tag{22}$$

2.2.2 Asymptotic expansion of the incomplete Beta ratio function $I_x(a,b)$ for $a \gg b$

The aim is to find an asymptotic expansion of the beta ratio function $I_x(a,b)$ when $a \gg b$. Following [Dom96], we transform the expression for $I_x(a,b)$ to obtain

$$\begin{aligned} I_x(a,b) &= \frac{1}{B(a,b)} \int_{-\log(x)}^{+\infty} e^{-at}(1-e^{-t})^{b-1} dt \\ &= \frac{1}{B(a,b)} \int_{-\log(x)}^{+\infty} e^{-\left[a+\frac{b-1}{2}\right]t} t^{b-1} \frac{t^{1-b} e^{-\frac{b-1}{2}t}}{(e^t-1)^{1-b}} dt. \end{aligned}$$

In this form we recognize in the integral the generating function of the generalized Bernouilli polynomials $B_k^{(a)}(x)$ (see [Luk69], page 18). We have thus

$$\begin{aligned} I_x(a,b) &\sim \frac{1}{B(a,b)} \sum_{k=0}^{+\infty} c_k \int_{-\log(x)}^{+\infty} e^{-\nu t} t^{b-1+2k} dt \\ &= \frac{1}{B(a,b)} \frac{1}{\nu^b} \sum_{k=0}^{+\infty} c_k \frac{\Gamma(b+2k)}{\nu^{2k}} Q(b+2k, -\nu \log(x)) \end{aligned}$$

where $\nu = a + (b-1)/2$ and

$$c_k = \frac{B_{2k}^{(1-b)}\left(\frac{1-b}{2}\right)}{(2k)!}.$$

The c_k can be computed using the recurrence formula

$$c_n = \frac{z}{(2n+1)!2^{2n}} + \frac{1}{n}\sum_{m=1}^{n-1} \frac{mz-(n-m)}{(2m+1)!2^{2m}} c_{n-m}$$

with $z = b-1$ (see [DM92]).

Writing $D = -\nu \log(x)$ and using the series (11) we get

$$\begin{aligned} I_x(a,b) &\sim \frac{1}{B(a,b)} \frac{Q(b,D)}{\nu^b} \sum_{k=0}^{+\infty} c_k \frac{\Gamma(b+2k)}{\nu^{2k}} \\ &+ \frac{1}{B(a,b)} \frac{e^{-D}D^{b-1}}{\Gamma(b)} \sum_{k=1}^{+\infty} c_k \Gamma(b+2k) \left(\frac{1}{\nu}\right)^{2k} a_k(D) \end{aligned}$$

where the coefficients $a_k(D)$ are given by

$$a_k(D) = \sum_{j=1}^{2k} \frac{\Gamma(b)}{\Gamma(b+j)} D^j.$$

The asymptotic expansion of the Beta function $B(a,b)$ is (see [Dom96])

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sim \frac{1}{\nu^b} \sum_{k=0}^{+\infty} c_k \Gamma(b+2k) \left(\frac{1}{\nu}\right)^{2k}$$

and thus

$$I_x(a,b) \sim Q(b,D) + \frac{e^{-D}D^{b-1}}{\Gamma(b)} \left[\frac{\sum_{k=1}^{+\infty} c_k \Gamma(b+2k)\nu^{-2k} a_k(D)}{\sum_{k=0}^{+\infty} c_k \Gamma(b+2k)\nu^{-2k}} \right]$$

2.2.3 Expansion of the incomplete Beta ratio function $I_x(a+1, b+1)$

The aim is to develop a new expansion for the incomplete Beta ratio function using the 2.1.1.

Using the same notations for p and q, and the change of variable defined in 2.1.1, I_x appears in the form

$$I_x(a+1,b+1) = \frac{p^a q^b}{B(a+1,b+1)} \sum_{k=0}^{\infty} (k+1) a_{k+1} \int_{-\infty}^{D} e^{-(a+b)\frac{u^2}{2}} u^k du.$$

with D defined by

$$\frac{D^2}{2} = -p\log(x/p) - q\log((1-x)/q).$$

Noting

$$\phi(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \qquad \Phi_n(z) = \int_{-\infty}^{z} u^n \phi(u) du$$

and $z = \sqrt{a+b}D$, we get

$$\begin{aligned} I_x(a+1,b+1) &= \frac{p^a q^b}{B(a+1,b+1)}\sqrt{\frac{2\pi}{a+b}}\sum_{k=0}^{\infty}\frac{(2k+1)a_{2k+1}}{(a+b)^k}\Phi_{2k}(z) \\ &+ \frac{p^a q^b}{B(a+1,b+1)}\frac{\sqrt{2\pi}}{a+b}\sum_{k=0}^{\infty}\frac{(2k+2)a_{2k+2}}{(a+b)^k}\Phi_{2k+1}(z). \end{aligned}$$

Using the relations (27) and the expansion of the Beta function computed in 2.1.1, we get

$$\begin{aligned} I_x(a+1,b+1) = \Phi_0(z) &- \phi(z)\frac{\sum_{k=1}^{+\infty}\frac{(2k+1)!!}{(a+b)^k}a_{2k+1}\left(\sum_{j=1}^{k}\frac{z^{2j-1}}{(2j-1)!!}\right)}{\sum_{k=0}^{+\infty}\frac{(2k+1)!!}{(a+b)^k}a_{2k+1}} \\ &- \frac{\phi(z)}{\sqrt{a+b}}\frac{\sum_{k=0}^{+\infty}\frac{(2k+2)!!}{(a+b)^k}a_{2k+2}\left(\sum_{j=0}^{k}\frac{z^{2j}}{(2j)!!}\right)}{\sum_{k=0}^{+\infty}\frac{(2k+1)!!}{(a+b)^k}a_{2k+1}}. \end{aligned}$$

A Some Results about the Gaussian Function

In various part of this document we will make use of the probability density function (pdf) of the standard normal distribution (the Gaussian function)

$$\phi(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \tag{23}$$

and of the cumulative distribution function (cdf) of the probability distribution evaluated at a number z

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left(-\frac{u^2}{2}\right)du, \quad z \in \mathbb{R}. \tag{24}$$

The moments of the normal distribution are given by

$$\int_{-\infty}^{+\infty} u^k \phi(u)du = \begin{cases} 0 & \text{if } k = 2n+1, \\ 1.3.\ldots(2n-1) = (2n-1)!! & \text{if } k = 2n. \end{cases} \tag{25}$$

We will also denote

$$\Phi_k(z) = \int_{-\infty}^{z} u^k \phi(u)du, \quad z \in \mathbb{R} \tag{26}$$

and use the relations, true for any $k \geq 1$

$$\Phi_k(z) = \begin{cases} -\phi(z)(2n)!!\sum_{j=0}^{n}\frac{z^{2j}}{(2j)!!} & \text{if } k = 2n+1, \\ (2n-1)!!\left(\Phi(z) - \phi(z)\sum_{j=1}^{n}\frac{z^{2j-1}}{(2j-1)!!}\right) & \text{if } k = 2n. \end{cases} \tag{27}$$

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